Summary for 9231 Further Pure Mathematics 1

Roots of polynomial equations

For this chapter, S_n refers to the sum of each root raised to the power of n.

- For quadratic equations of the form $ax^2 + bx + c = 0$, with roots α and β,

$$\sum \alpha = \alpha + \beta = -\frac{b}{a}$$
$$\sum \alpha\beta = \alpha\beta = +\frac{c}{a}$$

- For cubic equations of the form $ax^3 + bx^2 + cx + d = 0$ with roots α, β and γ,

$$\sum \alpha = \alpha + \beta + \gamma = -\frac{b}{a}$$
$$\sum \alpha\beta = \alpha\beta + \beta\gamma + \alpha\gamma = +\frac{c}{a}$$
$$\sum \alpha\beta\gamma = \alpha\beta\gamma = -\frac{d}{a}$$

- For quartic equations of the form $ax^4 + bx^3 + cx^2 + dx + e = 0$ with roots α, β, γ and δ,

$$\sum \alpha = \alpha + \beta + \gamma + \delta = -\frac{b}{a}$$
$$\sum \alpha\beta = \alpha\beta + \beta\gamma + \alpha\gamma + \beta\delta + \alpha\delta + \gamma\delta = +\frac{c}{a}$$
$$\sum \alpha\beta\gamma = \alpha\beta\gamma + \alpha\beta\delta + \beta\gamma\delta = -\frac{d}{a}$$
$$\sum \alpha\beta\gamma\delta = \alpha\beta\gamma\delta = +\frac{e}{a}$$

If we are asked to find S_3 of a cubic (S_1 can be found from the above formulae and S_2 from expansion), we can consider the following:

$$ax^3 + bx^2 + cx + d = 0$$

Now the roots of this equation are given to be α, β and γ. Therefore we can form the following the three equations:

$$a\alpha^3 + b\alpha^2 + c\alpha + d = 0$$

$$a\beta^3 + b\beta^2 + c\beta + d = 0$$

$$a\gamma^3 + b\gamma^2 + c\gamma + d = 0$$

We can add these together to form

$$a\gamma^3 + b\gamma^2 + c\gamma + d + a\beta^3 + b\beta^2 + c\beta + d + a\alpha^3 + b\alpha^2 + c\alpha + d = 0$$

$$a(\alpha^3 + \beta^3 + \gamma^3) + b(\alpha^2 + \beta^2 + \gamma^2) + c(\alpha + \beta + \gamma) + 3d = 0$$

$$aS_3 + bS_2 + cS_1 + 3d = 0$$

It should be evident from this example how this can be applied to other polynomials

To find S_n when n is larger than the degree of the polynomial, multiply both sides of the equation by x^N where N is an appropriate amount chosen such that S_n is found in the resulting equation.

Rational functions

For the rational function $y = \frac{P(x)}{Q(x)}$, this syllabus considers the follwing cases:

- $\deg(P(x))$ = $\deg(Q(x))$
- $\deg(P(x))$ + 1 = $\deg(Q(x))$
- $\deg(P(x))$ - 1 = $\deg(Q(x))$

Additionally, we will only work cases where the highest degree of either polynomial is 2.

When $\deg(P(x)) = \deg(Q(x)) \vee \deg(P(x)) + 1 = \deg(Q(x))$

In this case observe a horizontal and a vertical asymptote.

$y = \frac{P(x)}{Q(x)}$, a vertical asymptote is observed when $|y| \to \infty$. Now, $|y| \to \infty$ as $Q(x) \to 0$, so the vertical asymptote is found by solving $Q(x) = 0$. A horizontal asymptote is observed when $|x| \to \infty$, the value for y in this case can be found by writing $\frac{P(x)}{Q(x)}$ as partial fractions.

Turning points can be found by differentiation to infer at which points y has no value. Similarly, a quadratic in the form $P(x) - Q(x)y = 0$ can be used to identify the range for which y has no values (if such range exists) by setting its determinant to be less than 0.

The case where $\deg(P(x)) - 1 = \deg(Q(x))$

When long division is performed on $y = \frac{P(x)}{Q(x)}$ and y is obtained as partial fractions, one of the fractions will approach 0 as $x \to \infty$. The remaining is what serves as the equation of the line that is the oblique asymptote.

For this syllabus, $\frac{P(x)}{Q(x)}$ will be broken down into partial fractions in the form $Ax + B + \frac{C}{R(x)}$. As $x \to \infty$, $\frac{C}{R(x)} \to 0$, so the oblique asymptote is $y = Ax + B$. The vertical asymptote can be found, similar to the previous case, by $Q(x) = 0$.

Summation of series

Formula booklet has everything needed. Remember method of differences.

Matrices

An augmented matrix is created by combining the columns of two matrices. Row operations acting on both the original matrix and the identity matrix can be used to find the inverse of the original matrix. There are three types of row operations:

row switching	$r_i \leftrightarrow r_j$
row multiplication	$r_i \to k r_j$
row addition	$r_i \to r_i + k r_j$

A matrix with a determinant of 0 has no inverse. Such a matrix is called a **singular matrix**. A matrix that has an inverse is a **non-singular matrix**. You can also identify a singular matrix when you realise during row operations that no combination of row operations could turn it into an identity matrix.

Determinant of a 3x3 matrix

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

When multiple row multiplications (and row additions, considering k) are applied on a matrix, the determinant of the new matrix is the product of all the factors used in the row operations and the old determinant.

We can use this fact to first reduce a matrix to something easier to work with, calculate the determinant of this much simpler matrix then use that to find the orignal determinant.

Adjoint of a 3x3 matrix

To get each cell of the adjoint, hide the row and column of that cell in the original matrix, and get the determinant of the 2x2 matrix formed by the remaining numbers. The determinant is the value of that cell in the adjoint matrix.

Inverse of a 3x3 matrix

The inverse of a 3x3 matrix is its adjoint divided by its determinant. If the determinant is 0, the inverse is obviously undefined.

Matrix transformations

Transformation	**Matrix**
Stretch by a factor k in the x-direction	$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$
Stretch by a factor k in the y-direction	$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$
Enlargement with centre at the origin, scale factor k	$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$
Reflection in the x-axis	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
Reflection in the y-axis	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
Reflection in the line $y = x$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
Rotation about the origin by θ anticlockwise	$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Points unchanged by a transformation are called invariant points. The origin is always an invariant point. Invariant points for matrix $\mathbf{A}$ can be found with the following equation:

$$\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Invariant lines can be found with the following equation:

$$\mathbf{A}\begin{pmatrix} t \\ mt \end{pmatrix} = \begin{pmatrix} T \\ mT \end{pmatrix}$$

To find m, the gradient of the invariant line solve the simultaneous equations that result from considering the parametric equations. The y-intercept for the invariant lines is always 0 in linear transformations as they do not shift the origin.

Polar coordinates

In the polar coordinate system, points are represented by their anticlockwise angle and distance from the origin.

For all polar curves, $x = r\cos\theta$, $y = r\sin\theta$, and $x^2 + y^2 = r^2 \Rightarrow r \geq 0$. The function is given in the form $r = f(\theta)$.

Parametric polar form is $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$.

Curve sketching

- Use a table to work out values of r for different values of θ.
- Determine, if any, the values of θ when $r = 0$.
- Make full use of curve symmetry and lines of symmetry; for example, cosine-based curves have symmetry about the line $\theta = 0$

Spirals in this syllabus have the forms $r = a\theta$, $r = a\mathrm{e}^{k\theta}$, and $r = \frac{a}{\theta}$. Cardioids have the form $r = a + b\cos\theta$.

Area

The formula for the area of a polar curve is $A = \frac{1}{2}\int r^2 \,\mathrm{d}\theta$. If the limits of two curves differ, their integrals cannot be combined.

Maxima and minima

- For the greatest distance from the origin, use $\frac{\mathrm{d}r}{\mathrm{d}\theta}$.
- For the maximum distance from the line $\theta = 0$, use $\frac{\mathrm{d}y}{\mathrm{d}\theta}$.
- For the maximum or minimum distance from the line $\theta = \frac{\pi}{2}$, use $\frac{\mathrm{d}x}{\mathrm{d}\theta}$

Vectors

Vector product

The vector product of $\vec{a}$ and $\vec{b}$ is $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \hat{n}$ where $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$.

The area of a triangle OAB is given as $\frac{1}{2}|\vec{a} \times \vec{b}|$.

Vector equation of a line

The shortest distance between a point and a line is derived from $\overrightarrow{PQ} \cdot \vec{b} = 0$, where P is the point and Q is any chosen point on the line.

The shortest distance between skew lines is

$$\left| \frac{(\vec{a_1} - \vec{a_2}) \cdot \left(\vec{b_1} \times \vec{b_2}\right)}{\left|\vec{b_1} \times \vec{b_2}\right|} \right|$$

where a_1 and a_2 are points on the lines, and b_1 and b_2 are the directions of the lines.

Planes

Equation type	Form
Scalar	$\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$
Cartesian	$ax + by + cz = d$
Vector	$\vec{r} = \vec{a} + \vec{b}s + \vec{c}t$

The angle between two planes can be found by comparing the angle between their normals. The same can be said of the angle between a plane and a line.

To find the line of intersection between two planes, we set one of the variables as a free variable, then obtain a set of equations from the vector equation.

To show that a line lies on a plane, substitute the parametric equations of the line into the Cartesian equation of the plane.

For a line meeting the plane, substitute its paramtric equations into the Cartesian equation of the plane to find at which point they intersect.

For the shortest distance between the point P and the plane $\vec{r} \cdot \vec{n} = d$ use $\left|\frac{\overline{PQ} \cdot \vec{n}}{|\vec{n}|}\right|$, where Q is any chosen point on the plane.

Proof by induction

To prove something by induction, form an appropriate proposition P_k, Then show that $P_k \Rightarrow P_{k+1}$. Then, show that P_1 is true. Put them together to show that P_n is true for $n \in \mathbb{N}$.

In this syllabus we will be seeing proofs in the form:

$$(P_k \Rightarrow P_{k+1}) \wedge P_1 \therefore P_n \text{ for } n \in \mathbb{N}$$